Kuroda's Translation for Higher-Order Logic

Logimics Meeting

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Brouwer–Hilbert controversy

- For L.E.J. Brouwer, proofs must be **constructive**
  – A proof of $\exists x.A$ must give a witness x
  – A proof of $A \vee B$ must either prove A or prove B

- Intuitionism: **reject** the principle of excluded middle $A \vee \neg A$

- For David Hilbert, the principle of excluded middle is fundamental

"Taking the principle of excluded middle from the mathematician would be the same, say, as proscribing the telescope to the astronomer or to the boxer the use of his fists"

Intuitionistic logic

- **Intuitionistic** logic = classical logic without the principle of excluded middle $A \vee \neg A$

- Drawbacks:
  - No double-negation elimination $\neg\neg A \Rightarrow A$
  - No proof by contradiction

$$\frac{\Gamma, \neg A \vdash \bot}{\Gamma \vdash A}$$

- Advantage: **constructive** proofs

- What is the link between provability in classical logic and provability in intuitionistic logic?

Embedding classical logic into intuitionistic logic

- **Glivenko's theorem** [1928]
  In propositional logic, we have $\underbrace{\Gamma \vdash_c A}_{\text{classical logic}}$ iff $\underbrace{\neg\neg\Gamma \vdash_i \neg\neg A}_{\text{intuitionistic logic}}$

- Intuition:

$$\vdash_i A \vee \neg A \quad ✗$$

$$\vdash_i \neg\neg(A \vee \neg A) \quad ✓$$

- What about first-order logic?

Double-negation translations (1)

- Translations inserting **double negations** inside formulas
  - Kolmogorov [1925]
  - Gödel [1933] and Gentzen [1936]
  - Kuroda [1951]

- Kuroda's translation: $A^{Ku} := \neg\neg A_{Ku}$ and

$$
\begin{array}{lll}
(A \Rightarrow B)_{Ku} := A_{Ku} \Rightarrow B_{Ku} & (\neg A)_{Ku} := \neg A_{Ku} & P_{Ku} := P \text{ if } P \text{ atomic} \\
(A \wedge B)_{Ku} := A_{Ku} \wedge B_{Ku} & \top_{Ku} := \top & (\forall x.A)_{Ku} := \forall x.\neg\neg A_{Ku} \\
(A \vee B)_{Ku} := A_{Ku} \vee B_{Ku} & \bot_{Ku} := \bot & (\exists x.A)_{Ku} := \exists x.A_{Ku}
\end{array}
$$

Double-negation translations (2)

- Translations $A \mapsto A^t$ that satisfy:

  Property (1) if $\Gamma \vdash_c A$ then $\Gamma^t \vdash_i A^t$

  Property (2) $\vdash_c A^t \Leftrightarrow A$

- Using Property (2):

  if $\Gamma^t \vdash_i A^t$ then $\Gamma \vdash_c A$

- What about higher-order logic?

Higher-order logic

- It is possible to quantify over propositions in higher-order logic
- Syntax of simple type theory

$$\begin{array}{rll} t, u ::= & x & \text{Variables} \\ & \mid c & \text{Constants} \\ & \mid tu & \text{Applications} \\ & \mid \lambda x.t & \lambda\text{-abstractions} \end{array}$$

- **Computation**
  - β-reduction: $(\lambda x.t)u \hookrightarrow t[x \leftarrow u]$
  - Congruence $\equiv_\beta$

Inference rules (1)

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\ \textsc{Imp-I} \qquad \frac{\Gamma \vdash A \Rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B}\ \textsc{Imp-E}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \textsc{And-I} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ \textsc{And-EL} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ \textsc{And-ER}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ \textsc{Or-IL} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ \textsc{Or-IR} \qquad \frac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C}\ \textsc{Or-E}$$

$$\frac{\Gamma, A \vdash \bot}{\Gamma \vdash \neg A}\ \textsc{Not-I} \qquad \frac{\Gamma \vdash \neg A \qquad \Gamma \vdash A}{\Gamma \vdash \bot}\ \textsc{Not-E}$$

Inference rules (2)

$$\frac{\Gamma \vdash A \qquad x \notin FV(\Gamma)}{\Gamma \vdash \forall x.A}\ \textsc{All-I} \qquad \frac{\Gamma \vdash \forall x.A}{\Gamma \vdash A[x \leftarrow t]}\ \textsc{All-E}$$

$$\frac{\Gamma \vdash A[x \leftarrow t]}{\Gamma \vdash \exists x.A}\ \textsc{Ex-I} \qquad \frac{\Gamma \vdash \exists x.A \qquad \Gamma, A \vdash C \qquad x \notin FV(\Gamma, C)}{\Gamma \vdash C}\ \textsc{Ex-E}$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A}\ \textsc{Bot-E} \qquad \frac{}{\Gamma \vdash \top}\ \textsc{Top-I}$$

$$\frac{\Gamma \vdash A \qquad A \equiv_\beta B}{\Gamma \vdash B}\ \textsc{Conv} \qquad \frac{}{\Gamma \vdash A \lor \neg A}\ \mathrm{PEM}$$

Higher-order logic with equality

- Introduction and elimination rules for the new symbol $=$

$$\frac{}{\Gamma \vdash u = u}\ \textsc{Eq-I} \qquad\qquad \frac{\Gamma \vdash A[x \leftarrow u] \qquad \Gamma \vdash u = v}{\Gamma \vdash A[x \leftarrow v]}\ \textsc{Eq-E}$$

- **Functional extensionality**: two point-wise equal functions are equal

$$\frac{\Gamma \vdash fx = gx \qquad x \notin FV(\Gamma, f, g)}{\Gamma \vdash f = g}\ \textsc{FunExt}$$

- **Propositional extensionality**: two equivalent propositions are equal

$$\frac{\Gamma \vdash A \Rightarrow B \qquad \Gamma \vdash B \Rightarrow A}{\Gamma \vdash A = B}\ \textsc{PropExt}$$

Notations

- Higher-order logic **without** equality

$$\Gamma \vdash_i A \text{ and } \Gamma \vdash_c A$$

- Higher-order logic **with** equality

$$\Gamma \vdash_i^* A \text{ and } \Gamma \vdash_c^* A \text{ with } * \in \{\mathfrak{e}, \mathfrak{ep}, \mathfrak{ef}, \mathfrak{efp}\}$$

$\mathfrak{e}$ for Eq-I and Eq-E
$\mathfrak{p}$ for PropExt
$\mathfrak{f}$ for FunExt

Double-negation translations for higher-order logic

- Investigated by Brown and Rizkallah [2014]

- Kolmogorov's and Gödel-Gentzen's translations **cannot** be extended to higher-order logic

- Kuroda's translation can be extended, but
  – they did not prove **Property (2)**
  – they proved that Property (1) **fails** in the presence of functional extensionality

Contribution

- We prove that **functional extensionality** and **propositional extensionality** are sufficient to derive Property (2)
- We give a **condition** under which Property (1) holds with functional extensionality

Kuroda's translation for higher-order logic

- Inserting double negations in front of formulas

$$A^{Ku} := \neg\neg A_{Ku}$$

- Inserting double negations after universal quantifiers

A_{Ku} is inductively defined by:

$$
\begin{array}{lcl}
x_{Ku} & := & x \\
c_{Ku} & := & \begin{cases} \lambda p.\forall x.\neg\neg(px) & \text{if } c = \forall \\ c & \text{otherwise} \end{cases} \\
(\lambda x.t)_{Ku} & := & \lambda x.t_{Ku} \\
(tu)_{Ku} & := & t_{Ku}u_{Ku}
\end{array}
$$

Key results

- We do not have $(A[z \leftarrow w])^{Ku} = A^{Ku}[z \leftarrow w]$ anymore
- Substitution

$$(A[z \leftarrow w])^{Ku} = A^{Ku}[z \leftarrow w_{Ku}]$$

- Conversion

$$\text{if } A \equiv_\beta B \text{ then } A^{Ku} \equiv_\beta B^{Ku}$$

Property (1)

- In higher-order logic:

  1. If $\Gamma \vdash_c A$ then $\Gamma^{Ku} \vdash_i A^{Ku}$
  2. For $* \in \{\mathfrak{e}, \mathfrak{ep}\}$, if $\Gamma \vdash_c^* A$ then $\Gamma^{Ku} \vdash_i^* A^{Ku}$
  3. For $* \in \{\mathfrak{ef}, \mathfrak{efp}\}$, if $\Gamma \vdash_c^* A$ then $\Delta_{eq}, \Gamma^{Ku} \vdash_i^* A^{Ku}$

- Double-negation elimination on equality Δ_{eq}: $\forall x \forall y. \neg\neg(x = y) \Rightarrow x = y$

Property (2)

- Not necessarily true
  - P of type o (proposition) and C constant of type $o \to o$
  - We have $\vdash_c (CP)^{Ku} \Leftrightarrow \neg\neg(CP_{Ku}) \Leftrightarrow CP_{Ku}$
  - We cannot derive $\vdash_c (CP)^{Ku} \Leftrightarrow CP$

- We assume **functional extensionality** and **propositional extensionality**

$$\vdash_c^{\mathfrak{efp}} A^{Ku} \Leftrightarrow A$$

Back to classical logic

Using Property (2)

1. If $\Gamma^{Ku} \vdash_i A^{Ku}$ then $\Gamma \vdash_c^{\mathfrak{efp}} A$
2. For $* \in \{\mathfrak{e}, \mathfrak{ep}\}$, if $\Gamma^{Ku} \vdash_i^* A^{Ku}$ then $\Gamma \vdash_c^{\mathfrak{efp}} A$
3. For $* \in \{\mathfrak{ef}, \mathfrak{efp}\}$, if $\Delta_{eq}, \Gamma^{Ku} \vdash_i^* A^{Ku}$ then $\Gamma \vdash_c^{\mathfrak{efp}} A$

Takeaway message

- Kuroda's translation extends to higher-order logic
- In the **absence** of functional extensionality, Property (1) holds [Brown, Rizkallah, 2014]
- In the **presence** of functional extensionality, Property (1) holds when assuming the double-negation elimination on equality
- Property (2) holds when assuming both **functional extensionality** and **propositional extensionality**