Kuroda's Translation for Higher-Order Logic

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Intuitionistic logic

- **Intuitionistic** logic = classical logic without the principle of excluded middle $A \vee \neg A$

- Drawbacks:
  - No double-negation elimination $\neg\neg A \Rightarrow A$
  - No proof by contradiction

$$\frac{\Gamma, \neg A \vdash \bot}{\Gamma \vdash A}$$

- Advantage: **constructive** proofs

- What is the link between provability in classical logic and provability in intuitionistic logic?

Embedding classical logic into intuitionistic logic

- **Glivenko's theorem** [1928]
  In propositional logic, we have $\underbrace{\Gamma \vdash_c A}_{\text{classical logic}}$ iff $\underbrace{\neg\neg\Gamma \vdash_i \neg\neg A}_{\text{intuitionistic logic}}$

- Intuition:

$$\vdash_i A \vee \neg A \quad ✗$$

$$\vdash_i \neg\neg(A \vee \neg A) \quad ✓$$

- What about first-order logic?

Double-negation translations

- Translations $A \mapsto A^t$ that satisfy:

  Property (1) if $\Gamma \vdash_c A$ then $\Gamma^t \vdash_i A^t$

  Property (2) $\vdash_c A^t \Leftrightarrow A$

  Using Property (2): if $\Gamma^t \vdash_i A^t$ then $\Gamma \vdash_c A$

- Translations inserting **double negations** inside formulas
  – Examples: Kolmogorov [1925], Gödel-Gentzen [1933, 1936], Kuroda [1951]
  – Kuroda's translation inserts double negations in front of formulas and after $\forall$

- What about higher-order logic?

Higher-order logic

- Syntax $t, u ::= x \mid c \mid tu \mid \lambda x.t$

- Logical constants: $\wedge, \vee, \Rightarrow, \neg, \forall, \exists, \top, \bot$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\ \textsc{Imp-I} \qquad \frac{\Gamma \vdash A \Rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B}\ \textsc{Imp-E} \qquad \dots$$

- **Computation**
  - β-reduction: $(\lambda x.t)u \hookrightarrow t[x \leftarrow u]$
  - Additional inference rule:

$$\frac{\Gamma \vdash A \qquad A \equiv_\beta B}{\Gamma \vdash B}\ \textsc{Conv}$$

Higher-order logic with equality

- Inference rules for the new symbol $=$

$$\frac{}{\Gamma \vdash u = u}\ \textsc{Eq-I} \qquad \frac{\Gamma \vdash A[x \leftarrow u] \qquad \Gamma \vdash u = v}{\Gamma \vdash A[x \leftarrow v]}\ \textsc{Eq-E}$$

$$\frac{\Gamma \vdash fx = gx \qquad x \notin FV(\Gamma, f, g)}{\Gamma \vdash f = g}\ \textsc{FunExt} \qquad \frac{\Gamma \vdash A \Rightarrow B \qquad \Gamma \vdash B \Rightarrow A}{\Gamma \vdash A = B}\ \textsc{PropExt}$$

- We write $\Gamma \vdash_i^* A$ and $\Gamma \vdash_c^* A$ with $* \in \{\mathfrak{e}, \mathfrak{ep}, \mathfrak{ef}, \mathfrak{efp}\}$
  $\mathfrak{e}$ for $\textsc{Eq-I}$ and $\textsc{Eq-E}$
  $\mathfrak{p}$ for $\textsc{PropExt}$
  $\mathfrak{f}$ for $\textsc{FunExt}$

Double-negation translations for higher-order logic

- Investigated by Brown and Rizkallah [2014]
- Kolmogorov's and Gödel-Gentzen's translations **cannot** be extended to higher-order logic
- Kuroda's translation can be extended, but
  – they did not prove **Property (2)**
  – they proved that Property (1) fails in the presence of functional extensionality

Contribution

- We prove that **functional extensionality** and **propositional extensionality** are sufficient to derive Property (2)
- We give a **condition** under which Property (1) holds with functional extensionality

Kuroda's translation for higher-order logic

- A_{Ku} is inductively defined by:

$$
\begin{array}{lcl}
x_{Ku} & := & x \\
c_{Ku} & := & \begin{cases} \lambda p.\forall x.\neg\neg(px) & \text{if } c = \forall \\ c & \text{otherwise} \end{cases} \\
(\lambda x.t)_{Ku} & := & \lambda x.t_{Ku} \\
(tu)_{Ku} & := & t_{Ku} u_{Ku}
\end{array}
$$

  and $A^{Ku} := \neg\neg A_{Ku}$

- Substitution: $(A[z \leftarrow w])^{Ku} = A^{Ku}[z \leftarrow w_{Ku}]$

- Conversion: if $A \equiv_\beta B$ then $A^{Ku} \equiv_\beta B^{Ku}$

Property (1)

- In higher-order logic:

  1. If $\Gamma \vdash_c A$ then $\Gamma^{Ku} \vdash_i A^{Ku}$
  2. For $* \in \{\mathfrak{e}, \mathfrak{ep}\}$, if $\Gamma \vdash_c^* A$ then $\Gamma^{Ku} \vdash_i^* A^{Ku}$
  3. For $* \in \{\mathfrak{ef}, \mathfrak{efp}\}$, if $\Gamma \vdash_c^* A$ then $\Delta_{eq}, \Gamma^{Ku} \vdash_i^* A^{Ku}$

- Double-negation elimination on equality Δ_{eq}: $\forall x \forall y. \neg\neg(x = y) \Rightarrow x = y$

Property (2)

- Not necessarily true
  - P of type o (proposition) and C constant of type $o \to o$
  - We have $\vdash_c (CP)^{Ku} \Leftrightarrow \neg\neg(CP_{Ku}) \Leftrightarrow CP_{Ku}$
  - We cannot derive $\vdash_c (CP)^{Ku} \Leftrightarrow CP$ without further assumptions

- We assume **functional extensionality** and **propositional extensionality**

$$\vdash_c^{\mathfrak{efp}} A^{Ku} \Leftrightarrow A$$

Using Property (2)

1. If $\Gamma^{Ku} \vdash_i A^{Ku}$ then $\Gamma \vdash_c^{\mathfrak{efp}} A$
2. For $* \in \{\mathfrak{e}, \mathfrak{ep}\}$, if $\Gamma^{Ku} \vdash_i^* A^{Ku}$ then $\Gamma \vdash_c^{\mathfrak{efp}} A$
3. For $* \in \{\mathfrak{ef}, \mathfrak{efp}\}$, if $\Delta_{eq}, \Gamma^{Ku} \vdash_i^* A^{Ku}$ then $\Gamma \vdash_c^{\mathfrak{efp}} A$

Takeaway message

- Kuroda's translation extends to higher-order logic
- In the **absence** of functional extensionality, Property (1) holds [Brown, Rizkallah, 2014]
- In the **presence** of functional extensionality, Property (1) holds when assuming the double-negation elimination on equality
- Property (2) holds when assuming both **functional extensionality** and **propositional extensionality**