

Connecting Higher-Order Logics using Kolmogorov's Translation à la Dowek-Werner

Tribute to Gilles Dowek

Thomas Traversié



Connections between Logics (1)

- Different logics:

- ▶ **Minimal logic** (ML)
- ▶ **Intuitionistic logic** (IL) = ML + principle of explosion $\perp \Rightarrow A$
- ▶ **Classical logic** (CL) = CL + principle of excluded middle $A \vee \neg A$

Connections between Logics (1)

- Different logics:

- ▶ **Minimal logic** (ML)
- ▶ **Intuitionistic logic** (IL) = ML + principle of explosion $\perp \Rightarrow A$
- ▶ **Classical logic** (CL) = CL + principle of excluded middle $A \vee \neg A$

- Glivenko's theorem [1928] for **propositional logic**

- ▶ If $\vdash_{\text{CL}} A$ then $\vdash_{\text{IL}} \neg\neg A$
- ▶ Intuition: $\neg\neg(A \vee \neg A)$ has an intuitionistic proof

Connections between Logics (2)

- Double-negation translations for **first-order logic**
 - ▶ Insert double negations inside formulas
 - ▶ Several translations: Kolmogorov [1925], Gödel-Gentzen [1933, 1936], Kuroda [1951]
 - ▶ Embed CL into IL/ML

Connections between Logics (2)

- Double-negation translations for **first-order logic**
 - ▶ Insert double negations inside formulas
 - ▶ Several translations: Kolmogorov [1925], Gödel-Gentzen [1933, 1936], Kuroda [1951]
 - ▶ Embed CL into IL/ML
- Work begun supervised by Gilles: investigating these connections for **higher-order logic**

Higher-Order Logic

- Modeled using simple type theory
 - ▶ Terms: $t, u ::= x \mid c \mid t \ u \mid \lambda x. t$
 - ▶ Types: $\tau, \sigma ::= o \mid \iota \mid \tau \rightarrow \sigma$

Higher-Order Logic

- Modeled using simple type theory

- Terms: $t, u ::= x \mid c \mid t \ u \mid \lambda x. t$

- Types: $\tau, \sigma ::= o \mid \iota \mid \tau \rightarrow \sigma$

- Logical constants with the usual natural deduction inference rules

- $\wedge, \vee, \Rightarrow, \perp$

- $\forall_\tau, \exists_\tau$

- $\neg A := A \Rightarrow \perp$

Higher-Order Logic

- Modeled using simple type theory

- ▶ Terms: $t, u ::= x \mid c \mid t \ u \mid \lambda x. t$
- ▶ Types: $\tau, \sigma ::= o \mid \iota \mid \tau \rightarrow \sigma$

- Logical constants with the usual natural deduction inference rules

- ▶ $\wedge, \vee, \Rightarrow, \perp$
- ▶ $\forall_\tau, \exists_\tau$
- ▶ $\neg A := A \Rightarrow \perp$

- **Conversion** relation $\equiv_\beta$

- ▶ Generated by β -reduction: $(\lambda x. t) \ u \hookrightarrow t[x \leftarrow u]$
- ▶ Additional inference rule:

$$\frac{\Gamma \vdash A \quad A \equiv_\beta B}{\Gamma \vdash B} \text{CONV}$$

Kolmogorov's Translation

- Inserts double negations in front of every **subformula**:

$$\begin{array}{ll} (A \Rightarrow B)^{\text{ko}} & := \neg\neg(A^{\text{ko}} \Rightarrow B^{\text{ko}}) & A^{\text{ko}} & := \neg\neg A \text{ if } A \text{ is atomic or } \perp \\ (A \wedge B)^{\text{ko}} & := \neg\neg(A^{\text{ko}} \wedge B^{\text{ko}}) & (\forall x.A)^{\text{ko}} & := \neg\neg\forall x.A^{\text{ko}} \\ (A \vee B)^{\text{ko}} & := \neg\neg(A^{\text{ko}} \vee B^{\text{ko}}) & (\exists x.A)^{\text{ko}} & := \neg\neg\exists x.A^{\text{ko}} \end{array}$$

- Cannot be **naturally** extended to higher-order logic [Brown-Rizkallah 2014]

Kolmogorov's Translation

- Inserts double negations in front of every **subformula**:

$$\begin{array}{ll} (A \Rightarrow B)^{\text{ko}} & := \neg\neg(A^{\text{ko}} \Rightarrow B^{\text{ko}}) & A^{\text{ko}} & := \neg\neg A \text{ if } A \text{ is atomic or } \perp \\ (A \wedge B)^{\text{ko}} & := \neg\neg(A^{\text{ko}} \wedge B^{\text{ko}}) & (\forall x.A)^{\text{ko}} & := \neg\neg\forall x.A^{\text{ko}} \\ (A \vee B)^{\text{ko}} & := \neg\neg(A^{\text{ko}} \vee B^{\text{ko}}) & (\exists x.A)^{\text{ko}} & := \neg\neg\exists x.A^{\text{ko}} \end{array}$$

- Cannot be **naturally** extended to higher-order logic [Brown-Rizkallah 2014]
 - It does not preserve β -conversion: $(\lambda p. p) q \equiv_{\beta} q$ but

$$\begin{array}{ll} ((\lambda p. p) q)^{\text{ko}} & := (\lambda p. \neg\neg p) \neg\neg q \equiv_{\beta} \neg\neg\neg\neg q \\ q^{\text{ko}} & := \neg\neg q \end{array}$$

Kuroda's Translation

- Inserts double negations:

- after universal quantifiers:

$$(A \Rightarrow B)_{ku} := A_{ku} \Rightarrow B_{ku}$$

$$(A \wedge B)_{ku} := A_{ku} \wedge B_{ku}$$

$$(A \vee B)_{ku} := A_{ku} \vee B_{ku}$$

$$A_{ku} := A \text{ if } A \text{ is atomic or } \perp$$

$$(\forall x.A)_{ku} := \forall x. \neg \neg A_{ku}$$

$$(\exists x.A)_{ku} := \exists x. A_{ku}$$

- in front of formula: $A^{ku} := \neg \neg A_{ku}$

Kuroda's Translation

- Inserts double negations:

- ▶ after universal quantifiers:

$$(A \Rightarrow B)_{ku} := A_{ku} \Rightarrow B_{ku}$$

$$(A \wedge B)_{ku} := A_{ku} \wedge B_{ku}$$

$$(A \vee B)_{ku} := A_{ku} \vee B_{ku}$$

$$A_{ku} := A \text{ if } A \text{ is atomic or } \perp$$

$$(\forall x. A)_{ku} := \forall x. \neg \neg A_{ku}$$

$$(\exists x. A)_{ku} := \exists x. A_{ku}$$

- ▶ in front of formula: $A^{ku} := \neg \neg A_{ku}$

- Can be extended to higher-order logic [Brown-Rizkallah 2014, T. 2024]

- ▶ Preserves β -conversion

- ▶ Translates from higher-order CL to IL

Kuroda's Translation

- Inserts double negations:

- ▶ after universal quantifiers:

$$(A \Rightarrow B)_{ku} := A_{ku} \Rightarrow B_{ku}$$

$$(A \wedge B)_{ku} := A_{ku} \wedge B_{ku}$$

$$(A \vee B)_{ku} := A_{ku} \vee B_{ku}$$

$$A_{ku} := A \text{ if } A \text{ is atomic or } \perp$$

$$(\forall x.A)_{ku} := \forall x. \neg \neg A_{ku}$$

$$(\exists x.A)_{ku} := \exists x. A_{ku}$$

- ▶ in front of formula: $A^{ku} := \neg \neg A_{ku}$

- Can be extended to higher-order logic [Brown-Rizkallah 2014, T. 2024]

- ▶ Preserves β -conversion

- ▶ Translates from higher-order CL to IL

Key reason: treatment of atomic formulas in FOL/variables in HOL

Is it possible to extend Kolmogorov's translation to higher-order logic?

Is it possible to extend Kolmogorov's translation to higher-order logic?

The solution can be found in an old paper by Gilles!

Kolmogorov's Translation à la Dowek-Werner

■ Light Kolmogorov's translation [Dowek-Werner 2003]

$$(A \Rightarrow B)_{ko} := \neg\neg A_{ko} \Rightarrow \neg\neg B_{ko}$$

$$(A \wedge B)_{ko} := \neg\neg A_{ko} \wedge \neg\neg B_{ko}$$

$$(A \vee B)_{ko} := \neg\neg A_{ko} \vee \neg\neg B_{ko}$$

$$P_{ko} := P \text{ if } P \text{ atomic or } \perp$$

$$(\forall x. A)_{ko} := \forall x. \neg\neg A_{ko}$$

$$(\exists x. A)_{ko} := \exists x. \neg\neg A_{ko}$$

Kolmogorov's Translation à la Dowek-Werner

- Light Kolmogorov's translation [Dowek-Werner 2003]

$$(A \Rightarrow B)_{ko} := \neg\neg A_{ko} \Rightarrow \neg\neg B_{ko}$$

$$(A \wedge B)_{ko} := \neg\neg A_{ko} \wedge \neg\neg B_{ko}$$

$$(A \vee B)_{ko} := \neg\neg A_{ko} \vee \neg\neg B_{ko}$$

$$P_{ko} := P \text{ if } P \text{ atomic or } \perp$$

$$(\forall x.A)_{ko} := \forall x.\neg\neg A_{ko}$$

$$(\exists x.A)_{ko} := \exists x.\neg\neg A_{ko}$$

- Link: $A^{ko} = \neg\neg A_{ko}$

Kolmogorov's Translation à la Dowek-Werner

- Light Kolmogorov's translation [Dowek-Werner 2003]

$$(A \Rightarrow B)_{ko} := \neg\neg A_{ko} \Rightarrow \neg\neg B_{ko}$$

$$(A \wedge B)_{ko} := \neg\neg A_{ko} \wedge \neg\neg B_{ko}$$

$$(A \vee B)_{ko} := \neg\neg A_{ko} \vee \neg\neg B_{ko}$$

$$P_{ko} := P \text{ if } P \text{ atomic or } \perp$$

$$(\forall x.A)_{ko} := \forall x.\neg\neg A_{ko}$$

$$(\exists x.A)_{ko} := \exists x.\neg\neg A_{ko}$$

- Link: $A^{ko} = \neg\neg A_{ko}$

- Two versions of Kolmogorov's translation that generate the **same** result

Kolmogorov's Translation à la Dowek-Werner

- Light Kolmogorov's translation [Dowek-Werner 2003]

$$(A \Rightarrow B)_{ko} := \neg\neg A_{ko} \Rightarrow \neg\neg B_{ko}$$

$$(A \wedge B)_{ko} := \neg\neg A_{ko} \wedge \neg\neg B_{ko}$$

$$(A \vee B)_{ko} := \neg\neg A_{ko} \vee \neg\neg B_{ko}$$

$$P_{ko} := P \text{ if } P \text{ atomic or } \perp$$

$$(\forall x.A)_{ko} := \forall x.\neg\neg A_{ko}$$

$$(\exists x.A)_{ko} := \exists x.\neg\neg A_{ko}$$

- Link: $A^{ko} = \neg\neg A_{ko}$

- Two versions of Kolmogorov's translation that generate the **same** result

- **Two-step process** similar to Kuroda's translation

Kolmogorov-Dowek-Werner Translation for Higher-Order Logic

- Inserts double negations:

- ▶ on the arguments of the connectives and quantifiers:

$$\begin{aligned}x_{ko} &:= x \\c_{ko} &:= \begin{cases} \lambda p. \lambda q. c (\neg\neg p) (\neg\neg q) & \text{if } c \in \{\Rightarrow, \wedge, \vee\} \\ \lambda p. c (\lambda x. \neg\neg(p\ x)) & \text{if } c \in \{\forall, \exists\} \\ c & \text{otherwise} \end{cases} \\(\lambda x. t)_{ko} &:= \lambda x. t_{ko} \\(t\ u)_{ko} &:= t_{ko}\ u_{ko}\end{aligned}$$

- ▶ in front of the formula: $A^{ko} := \neg\neg A_{ko}$

Kolmogorov-Dowek-Werner Translation for Higher-Order Logic

- Inserts double negations:

- ▶ on the arguments of the connectives and quantifiers:

$$\begin{aligned}x_{ko} &:= x \\c_{ko} &:= \begin{cases} \lambda p. \lambda q. c (\neg\neg p) (\neg\neg q) & \text{if } c \in \{\Rightarrow, \wedge, \vee\} \\ \lambda p. c (\lambda x. \neg\neg(p\ x)) & \text{if } c \in \{\forall, \exists\} \\ c & \text{otherwise} \end{cases} \\(\lambda x. t)_{ko} &:= \lambda x. t_{ko} \\(t\ u)_{ko} &:= t_{ko}\ u_{ko}\end{aligned}$$

- ▶ in front of the formula: $A^{ko} := \neg\neg A_{ko}$

- Preserves β -conversion and translates from higher-order CL to ML

Conclusion

- Kolmogorov's translation
 - ▶ Original formulation: **cannot** be naturally extended to higher-order logic
 - ▶ Dowek-Werner formulation: **can** be naturally extended to higher-order logic

Conclusion

- Kolmogorov's translation
 - ▶ Original formulation: **cannot** be naturally extended to higher-order logic
 - ▶ Dowek-Werner formulation: **can** be naturally extended to higher-order logic
- Two-step processes can be extended to higher-order logic
 - ▶ **Kuroda's translation**: embeds higher-order CL into IL
 - ▶ **Kolmogorov's translation**: embeds higher-order CL into ML

Conclusion

- Kolmogorov's translation
 - ▶ Original formulation: **cannot** be naturally extended to higher-order logic
 - ▶ Dowek-Werner formulation: **can** be naturally extended to higher-order logic
- Two-step processes can be extended to higher-order logic
 - ▶ **Kuroda's translation**: embeds higher-order CL into IL
 - ▶ **Kolmogorov's translation**: embeds higher-order CL into ML
- Gödel-Gentzen translation
 - ▶ Cannot be naturally extended to higher-order logic [Brown-Rizkallah 2014]
 - ▶ Cannot be expressed by a two-step process

Last Words

- Gilles continued to work with us until the end

Last Words

- Gilles continued to work with us until the end
- He was a great advisor

Last Words

- Gilles continued to work with us until the end
- He was a great advisor

Thank you!